Role of pump diffraction on the stability of localized structures in degenerate optical parametric oscillators

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We show that the stability range of localized structures (LS's) in the form of minimum size phase domains in degenerate optical parametric oscillators is enhanced by increasing the diffraction of the pump wave. Pump diffraction enhances spatial oscillations of decaying tails of domain boundaries, whereas spatially oscillating (weakly decaying) tails prevent the collapse of LS's, enhance their stability range, and allow the existence of more complex LS's in the form of molecules.

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In degenerate optical parametric oscillators (DOPO's), two different kinds of localized structures (LS's) have been predicted so far. The first kind of LS's is related to the coexistence of the trivial zero solution and a modulationally unstable homogeneous solution. The LS's of this kind are characterized by a monotonic spatial decay of the field far from its center, and their shape is close to a sech(x) function [1]. The subcritical regime of DOPO operation is needed to excite such LS's.

The second kind of LS's is related to the coexistence of two homogeneous solutions, with the same amplitude but with phases differing by π. The LS's of this kind can be considered as the round phase domain of minimum size resting on the homogeneous background of the opposite phase or, equivalently, as round loops of domain walls (DW's). The DW's separating domains of opposite phase have been found in the form of isolated stripes [2,3] and rings of small radius [4–6], and were also experimentally observed in Ref. [7]. The DW's are characterized by an oscillatory spatial decay, which has been related with the stability of the LS [8].

In this paper we explore the spatial oscillatory decay of LS's in DOPO's, and prove that this decay has a strong effect on LS stability. We show that the spatial modulations strongly depend on the diffraction coefficient of the pump wave (on the ratio between diffraction coefficients of the fields), leading to an enhancement of the LS stability range when pump diffraction is increased.

We consider a doubly resonant DOPO, where both the subharmonics $A_1(r,t)$ and the pump wave $A_0(r,t)$ are close to a cavity resonance [9]

$$\partial_t A_1 = \gamma_1[-(1+i\Delta_1)A_1 + A_1^* A_0] + i a_1 \nabla^2 A_1, \qquad (1a)$$

$$\partial_t A_0 = \gamma_0[-(1+i\Delta_0)A_0 + E - A_1^2] + i a_0 \nabla^2 A_0, \qquad (1b)$$

where E is the amplitude of the (external) pumping field, Δ_1 and Δ_0 are the detunings of the resonators, γ_1 and γ_0 are the decay parameters, and a_1 and a_0 are the diffraction coefficients.

When the mirrors of the optical cavity are plane, the diffraction coefficients of signal and pump fields in a DOPO obey the relation $a_0/a_1 = 1/2$, as a consequence of the phase-matching condition [9]. On the other side, the curvature of the mirrors imposes a particular boundary condition not present in the model, which has been derived assuming the mean-field approximation. However, the use of resonators with curved mirrors close to a confocal (or more generally, self-imaging) configuration is nearly equivalent to the plane mirror case (with high Fresnel number and high level degeneracy of transverse modes), in which the diffraction coefficient depends on the deviation of the resonator length from confocality. In particular, the exactly confocal resonator is diffractionless (every ray has the same optical length in one round trip in the resonator). This equivalence has been shown analytically in Ref. [10], using a propagation matrix approach, and experimentally in Ref. [7], where the observed patterns were compared with the solutions of a mean-field model, with good agreement.

In the present case [Eqs.(1)], it is assumed that each field resonates in a near-self-imaging cavity, with different lengths. Then the diffraction coefficients can take independent values, their ratio being a parameter of the system. This configuration allows to explore the role that diffraction plays on the pattern formation properties in this system. In the following, we concentrate on the problem of the influence of diffraction on the stability of spatial localized structures.

The spatially homogeneous solution of Eq. (1) can be written as $\overline{A_1} = A\exp(i\varphi)$, $\overline{A_0} = (E - \overline{A_1^2})/(1+i\Delta_0)$, where [11]

$$A^2 = -1 + \Delta_1\Delta_0 + \sqrt{E^2 - (\Delta_1+\Delta_0)^2}, \qquad (2a)$$

$$\varphi = -\frac{1}{2}\arcsin\left(\frac{\Delta_1+\Delta_0}{E}\right), \qquad (2b)$$

with the additional constraint for the phase $\cos(2\varphi) > 0$.

Solution (2) can adopt two values, with the same amplitude but different phase values, differing by π. When spatial variations of the fields are allowed, both solutions can coexist, filling different regions in space and leading to a spatial distribution in the form of phase domains [4].

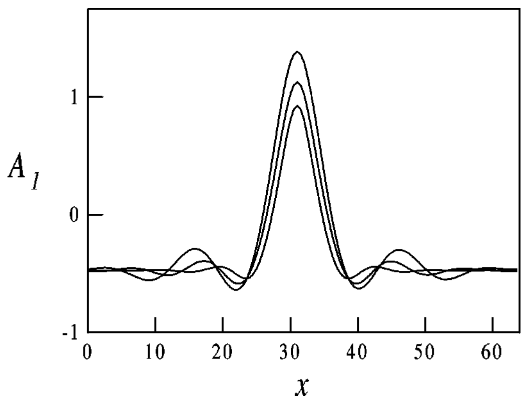

FIG. 1. Amplitude profile of a LS across a line crossing its center, evaluated numerically for different pump diffractions, $a_0 = 0.0005$, 0.002, and 0.01. The amplitude of the modulation of the tails increases with increasing diffraction. The other parameters are $a_1 = 0.001$, $E = 2$, $\Delta_1 = -0.6$, $\Delta_0 = 0$.

As shown in Ref. [4], the phase domains in DOPO can expand or contract depending on the value of signal detuning. For particular values of negative detuning the domains of arbitrary shape contract, but stop contracting at a minimum size, forming rotationally symmetric spatial localized structures (LS's). These LS's are characterized by a ring of zero intensity separating the states with different phase inside and outside the ring.

In the absence of modulational instabilities, the stability of these LS's depends on the mechanism opposing the contraction of domains. One opposing mechanism appears when a segment of a dark ring is repelled by the opposite segment, thus compensating the contraction. This mechanism describes well the formation of LS's in the Swift-Hohenberg equation [12], which describes the evolution of the signal field in DOPO operating close to the threshold [13].

Another mechanism is related to the fact that dark lines (or domain boundaries) do not decay monotonically but exhibit oscillations in space. A segment of the dark ring imposes a spatial oscillation of the intensity, and the opposite segment of the contracting ring can be fixed at an intensity minimum. In this way, the ring is stabilized due to spatial modulation, and consequently the stability of LS can be increased if the spatial oscillations are weakly damped.

This fact can be more clearly understood in the frame of the Swift-Hohenberg equation, which allows for a complementary description of the dynamics in terms of a potential functional. The interaction between the oscillating tails of the domain boundaries generates an oscillating potential, whose local minima correspond to localized structures of different orders at some ring radius [14]. The stronger the modulations are, the deeper the potential minima, thus enhancing the stability range.

Below we show that the physical mechanism responsible for the modulation of the tails in DOPO's is the diffraction of the pump wave at the transmission profile of the LS. In order to show this, we performed a numerical integration of DOPO equations (1) for different values of a_0. The amplitude along a line crossing the center of a LS is plotted in Fig. 1, showing that the diffraction strongly enhances the spatial oscillations.

The parameters defining the shape of a LS are the spatial decay and the wave number of the oscillating tails. They can

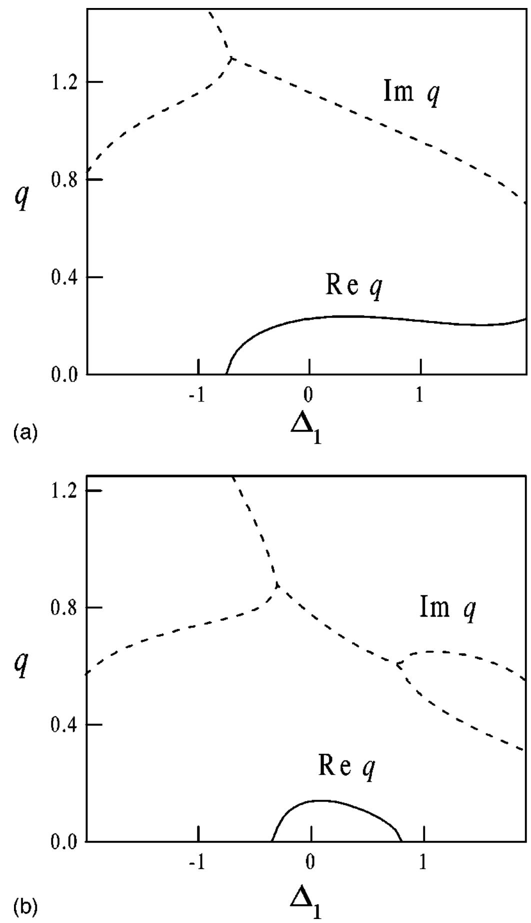

FIG. 2. Real part (continuous line) and imaginary part (dashed line) of the square root of the eigenvalues of the spatial stability analysis as a function of signal detuning, for two different values of the diffraction ratio: $a = 2$ (a) and $a = 10$ (b). The pump value is $E = 2.5$.

be analytically evaluated by means of a spatial stability analysis. Assume that the intensity of field is perturbed from its stationary value (2) in some place in transverse space (due to effects of boundaries, spatial perturbation, or defects in the patterns), and look how this perturbation decays (or grows) in space. For this purpose we consider evolution in space instead of time. When the system has reached a stationary state, the solutions, which we assume have radial symmetry, can be written in the time-independent form

$$A_i(r) = \overline{A_i} + \mathcal{A}_i(r). \tag{3}$$

After substituting Eq. (3) in Eq. (1), and considering regions in space not close to the domain boundary, the resulting system can be linearized in the deviations $\mathcal{A}_i(r)$, and the spatial evolution can be described by the system [15]

$$\nabla^2 \delta\mathbf{A} = \mathcal{L} \cdot \delta\mathbf{A}, \tag{4}$$

where $\delta\mathbf{A} = (\mathcal{A}_0, \mathcal{A}_0^*, \mathcal{A}_1, \mathcal{A}_1^*)^\dagger$ is a four-component perturbation vector and $\mathcal{L}$ is the linear matrix. In the case of resonant pump, $\Delta_0 = 0$, $\mathcal{L}$ is given by

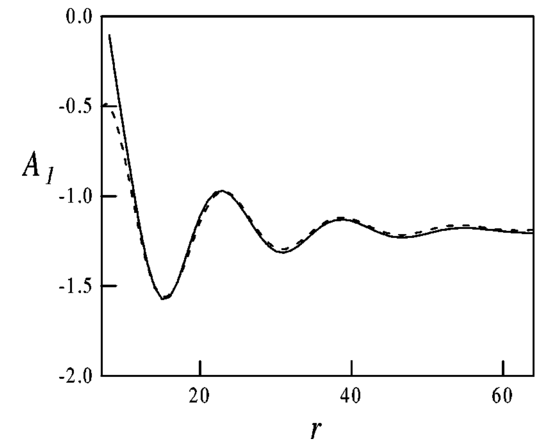

FIG. 3. Spatial oscillations of the field outside a LS, as evaluated numerically (continuous line) and analytically from the spatial stability analysis (dashed line), for E = 2.5, $\Delta_1=0.5$, $\Delta_0=0$, $a_1=0.00025$, $a_0=0.00125$ $(a=5)$.

$$\mathcal{L}=\begin{pmatrix} -i/a & 0 & -(2i/a)\overline{A_1} & 0 \\ 0 & i/a & 0 & (2i/a)\overline{A_1^*} \\ i\overline{A_1^*} & 0 & -i(1+i\Delta_1) & i\overline{A_0} \\ 0 & -i\overline{A_1} & -i\overline{A_0^*} & i(1-i\Delta_1) \end{pmatrix}. \quad (5)$$

The solutions of the linear system (4) are of the form

$$\delta\mathbf{A}(r)\propto e^{qr}, \quad (6)$$

where the wave vector is allowed to be complex, in the form $q=\mathrm{Re}\,q+i\,\mathrm{Im}\,q$. From Eq. (6), it follows that a negative value of $\mathrm{Re}\,q$ indicates a spatial decay of perturbation, and is responsible for the localization, while a nonvanishing value of $\mathrm{Im}\,q$ indicates the presence of a nonmonotonic (oscillatory) decay [16]. Thus, the solution (3), with the deviation given by Eq. (6), represents the assymptotic profile of the LS far from its core.

The expressions of spatial decay and modulation follow from the study of the eigenvalues μ of $\mathcal{L}$, which are the solutions of the characteristic equation

$$a^2\mu^4-2a^2\Delta_1\mu^3+(1-4aI_1)\mu^2-2\Delta_1(1-2aI_1)\mu + 4I_1(1+I_1)=0. \quad (7)$$

Comparing with the ansatz (6), we identify $q=\sqrt{\mu}$.

The simple analytical solution of Eq. (7) exists in the resonant signal case only, $\Delta_1=0$, being

$$a\mu^2=\frac{1}{\sqrt{2}}[-1+4aI_1\pm\sqrt{1-8a(2a+1)I_1}]^{1/2}, \quad (8)$$

where $I_1=E-1$.

We see from Eq. (8) that the size of the LS depends on the diffraction ratio a in a nontrivial way. This is in contrast with previous studies of pattern formation in many nonlinear optical systems (Lugiato-Lefever approach [17]), where diffraction appears simply as a scale factor in the wave vector, in the form ak^2.

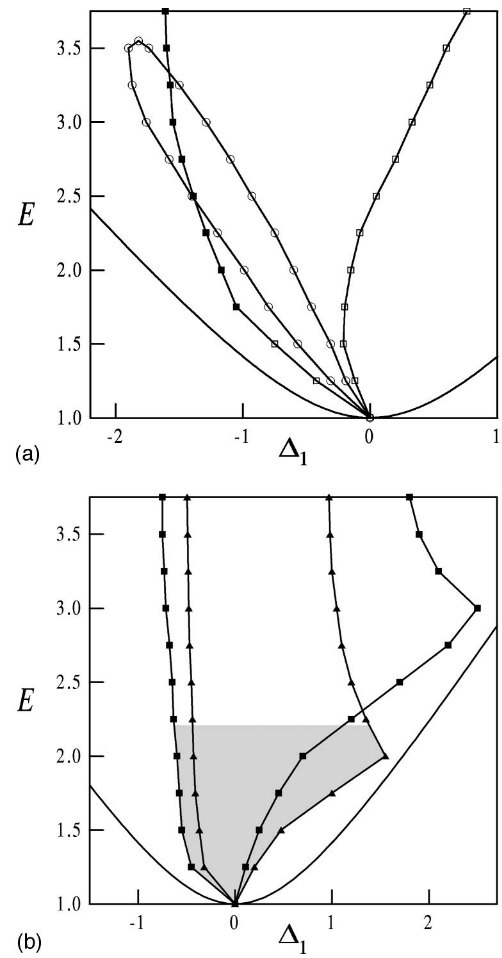

FIG. 4. Stability range of LS's, in the plane $\langle E,\Delta_1\rangle$, evaluated numerically for different values of diffraction ratio. In (a) the cases $a=0$ and $a=0.5$ are shown. In this case, the stability range is enhanced at any value of the pump. In (b), the cases $a=2$ and $a=5$ are shown. The stability range is enhanced now in the shadowed region. Above a critical pump, modulational instabilities decrease the stability range.

In Fig. 2 we show $\mathrm{Re}\,q$ (continuous line) and $\mathrm{Im}\,q$ (dashed line) evaluated from Eq. (7) as a function of signal detuning, for a fixed pump value $E=2.5$ and two diffraction values. In Fig. 2(a) the diffraction ratio is relatively small $(a=2)$. In this case, $\mathrm{Re}\,q$ is always nonzero for positive values of the detuning, and thus the spatial perturbations (due to the presence of DW's) decay for this particular value of the pump. For negative detunings, q can become imaginary. This corresponds to the ''off-resonance'' instability, in which roll patterns emerge.

For larger values of the diffraction ratio [see Fig. 2(b), where $a=10$], q becomes purely imaginary not only for negative, but also for positive values of the detuning, indicating the emergence of extended patterns in both sides of the resonance. This new modulational instability, induced by pump diffraction, has been discussed in Ref. [18], and leads to hexagonal patterns with different characteristics than those in the off-resonance region.

In order to check the validity of the previous results, Eqs. (1) were integrated numerically using a split-step algorithm on a square grid with periodic boundary conditions. Linear and nonlinear terms were solved in real space by means of a

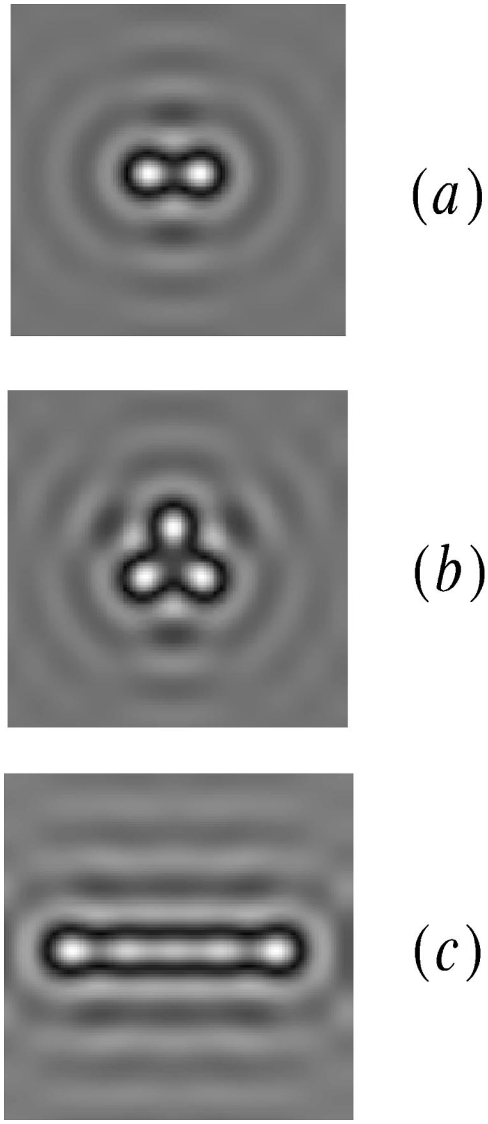

FIG. 5. Several bound states (molecules) of localized structures, obtained for $a=5, E=2.5$, $\Delta_0=0$, and $\Delta_1=0.5$. Double (a), triple (b) molecules, and a chain of five maxima (c) are shown.

Runge-Kutta routine, while nonlocal terms were solved in Fourier space with a fast Fourier transform code. We used typical grid sizes of 64×64, and a temporal step of order 10^{-3}. In every step a small amount of random noise was added in the signal field.

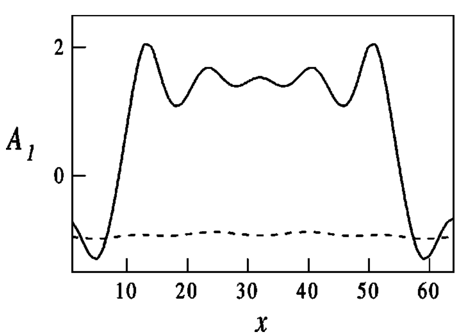

FIG. 6. Section across the y axis of the chain shown in Fig. 5(c), in the solid line. The dashed line represents a cut across across the space outside the dark line.

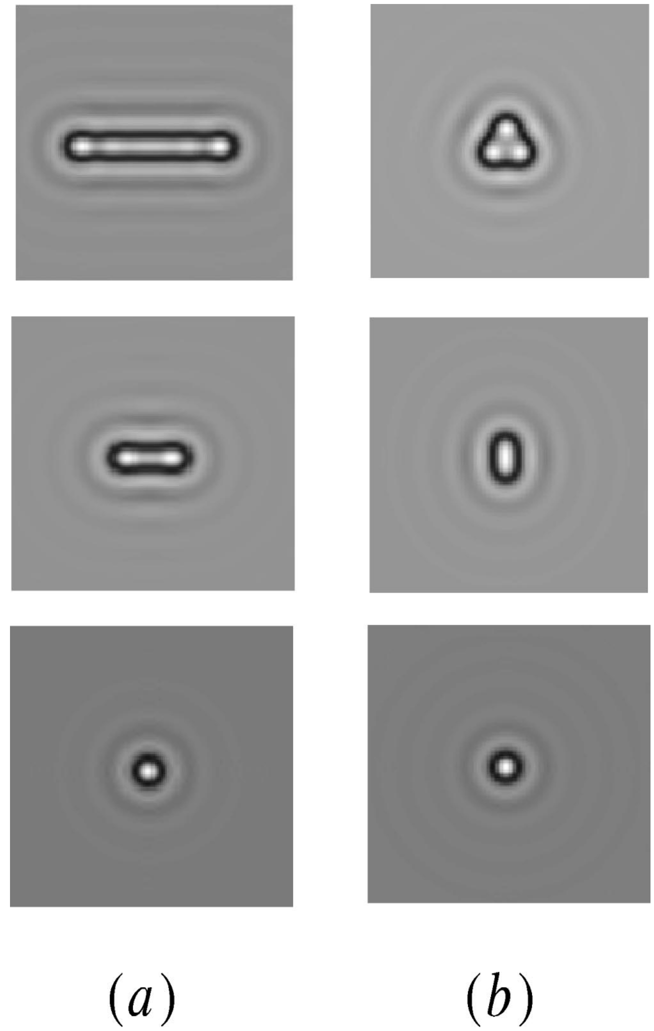

FIG. 7. Temporal evolution showing the decay to a single LS of the molecules given in Figs. 5(b) and 5(c), when the diffraction is decreased to $a=1$.

In Fig. 3, a comparison between analytical (dashed curve) and numerical results (continuous curve) for the spatial oscillations of the decaying tails of the domain boundary is given. The peak of the localized structure is omitted. Note that the correspondence is very good, even close to the domain boundary (line of zero intensity). The parameters are $E=2.5$, $\Delta_0=0$, $\Delta_1=0.5$, $a1=0.00025$, $a=5$. In this particular case four minima of the intensity are visible. The opposite segment of dark ring can be fixed by each of the minima. Obviously, the LS of the minimum size, as fixed by the first strongest maximum is the most stable one. However, dark rings with larger radii can also be stable [5].

The stability range of LS's is limited from one side due to contraction and annihilation of domains. From the other side the LS existence range is limited either due to the presence of modulational instabilities (modulations grow, and fill the whole space) or due to expansion of domains. Since the modulational instabilities are favored by diffraction, it may seem that it has a negative effect on the stability of LS's. However, for pump values at which instabilities are absent, it is expected that the increase in the modulation of the tails could prevent the full contraction, thus contributing to an enhancement of the stability range.

We have numerically calculated the stability range of LS for different values of pump diffraction (Fig. 4). In Fig. 4(a), the cases $a=0$ (circles) and $a=1/2$ (squares) are compared. Open (full) symbols indicates disappearance of LS's by collapse expansion of domains (by modulational instability). In this particular case, the stability of LS's is clearly enhanced for any value of the pump. The continuous curve denotes the existence range of the homogeneous solutions (2).

In Fig. 4(b) we represent the cases $a=2$ (squares) and $a=5$ (triangles). In this case, the stability range of LS's is enhanced in the shadowed region, so there is an upper limit for the pump value for the stability enhancement.

Our numerical calculations for larger pump diffractions show that the stability is always enhanced, at least up to some value of the pump.

The presence of strong modulations in the tails also allows the formation of more complex structures, in the form of bound states of single LS's, or molecules. Some examples of several molecules with different complexity are shown in Fig. 5. Examples with two and three maxima are shown in Figs. 5(a) and 5(b), and a chain composed of five maxima is shown in Fig. 5(c). The parameters chosen were $a=5$, $\Delta_1=-0.3$, $\Delta_0=0$, $E=2$ for all the pictures.

The internal structure of the chain given in Fig. 5(c) is more clearly appreciated in Fig. 6, where a cut across the middle ($y=32$) has been done (full line). The five maxima at equidistant points are seen. The field across a line outside the domain boundary is given by the dashed line, evaluated at $y=20$.

In all these cases, the large value of the pump diffraction is responsible for the stability of such complex structures, increasing the spatial oscillations and then preventing the collapse. To show this fact, we have followed the evolution of the molecules given in Figs. 5(b), 5(c), decreasing the diffraction value to $a=1$ and keeping the other parameters unchanged. The scenario is shown in Fig. 7, where the pictures have been taken at equally spaced times. The final state always corresponds to a single LS.

Concluding, we have investigated analytically the influence of the pump diffraction in the stability of LS's in a DOPO. The possibility of varying diffractions by means of the use of appropiate resonators allows us to show the important role that the amplitude oscillations of the spatial decay of a LS play in its stability. Analytical results based on a spatial stability analysis have been compared with the numerical integration of the model, with good agreement.

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